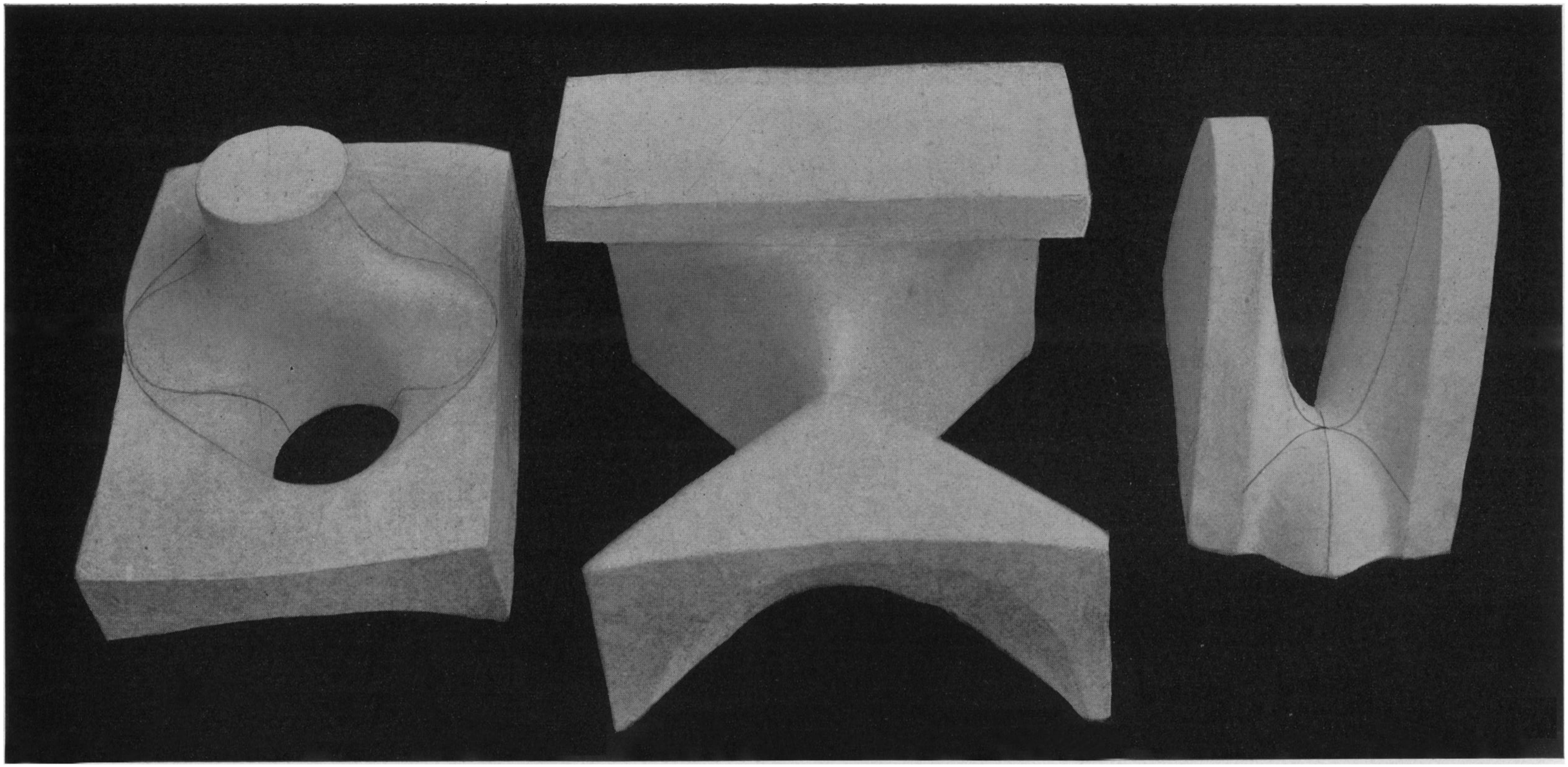

THE CARDIOID SURFACE

THE DELTOID SURFACE

TRANSLATION SURFACE
CONNECTED WITH THE CUSPIDAL CUBIC

On a Certain Class of Algebraic Translation-Surfaces.

BY JOHN EIESLAND.

In the following pages have been discussed types of translation-surfaces that are generated in four different ways. This class of surfaces owes its creation to S. Lie, who developed the general theory and also began a discussion of certain special cases. The important theorem by which he connected the parametric representation of such surfaces with the well-known theorem of Abel Lie always considered as one of his greatest achievements, and no one who reads his memoir in the Leipziger Berichte of 1892 can fail to admire the wonderful penetration of his genius. Two of Lie's students, R. Kummer and Georg Wiegner, have carried on the study of special cases, and Georg Scheffer, also one of his students, has in the Acta Mathematica, Vol. 24, given an admirable resumé of Lie's work and also an independent and elegant treatment of certain parts of the calculations. Recently Paul Stäckel, in the American Transactions, Vol. 7, has presented a paper on the minimal surfaces belonging to this class, in which the treatment is independent of Abel's theorem. It is to be hoped that his method will be fruitful in his further investigation along the same line. G. Scheffer, in the above mentioned paper, expresses his opinion that an extended and detailed treatment of this class of surfaces would be interesting not only from the standpoint of surface-theory, but also ought to be undertaken because of the bearing this subject has on the theory of functions.

As the field is very large and the number of types are very many, I have in this paper limited myself to algebraic surfaces only, and the following investigations will show that this limitation is not arbitrary, but is rather the most natural one in the discussion of the subject, inasmuch as I have shown that *the quartic curve which determines such surfaces must be unicursal and have no double points with distinct tangents.*

I have thought it advisable to introduce the subject by giving a short resumé of facts known long ago concerning translation-surfaces and statements of a few well known theorems.

I.

If a curve C_0, having a point p_0 in common with another curve C_1, is translated parallel to itself, any point on it will describe a curve C which is congruent with C_1 and similarly placed. The surface generated by C_0 may also be considered as generated by C and is called a *translation-surface.* It follows then that *a translation-surface may be generated in at least two different ways.*

The general parametric representation of such a surface is

$$x = A(u) + A_1(v), \quad y = B(u) + B_1(v), \quad z = C(u) + C_1(v), \tag{1}$$

whose generating curves are

$$x = A(u), \qquad y = B(u), \qquad z = C(u),$$

and

$$x = A_1(v), \qquad y = B_1(v), \qquad z = C_1(v).$$

A translation-surface may also be generated as follows: Consider the two curves

$$x = 2A(u), \qquad y = 2B(u), \qquad z = 2C(u),$$
$$x = 2A_1(v), \qquad y = 2B_1(v), \qquad z = 2C_1(v).$$

If we join any point on the first curve to a point on the second by a straight line, the locus of the middle points of this chord will evidently be the translation-surface

$$x = A(u) + A_1(v), \qquad y = B(u) + B_1(v), \qquad z = C(u) + C_1(v),$$

on which the generating curves are congruent to the given curves and similarly placed, but drawn on one-half the scale.* This definition is due to Lie.

Since the linear tangent at a point p on a (u) curve is moved parallel to itself, when this curve is translated along a (v) curve, the ensemble of all these parallel tangents forms a cylinder which is tangent to the surface, and hence, *the* (u) *and* (v) *lines are conjugate lines.*†

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* See Darboux, Leçons, Vol. I, p. 99.

† S. Lie, Beiträge zur Theorie der Minimalflächen (Math. Ann. t. XIV, pp. 332-337). See also Darboux, Leçons, Vol. I, p. 103.

If we construct the tangents to a curve C_1, we obtain a developable surface which cuts out a curve on the plane at infinity. Since all the curves C_1 are congruent and similarly placed, their respective developables will all cut out one and the same curve; the same holds for the curves C_0. Now suppose that to a tangent at a point on C_1 corresponds a parallel tangent at a point on C_0, or, what amounts to the same thing, the tangents to both curves are parallel to the elements of the same irreducible cone. In this case the two developables belonging to C_1 and C_0 respectively will cut out one and the same curve on the plane at infinity; moreover, *these curves have a common envelope*; for, since the tangents to the curves C_0 are parallel to each other at all points where they are cut by C_1, and since by hypothesis the tangents along C_1 and C_0 are parallel, there must exist on each curve C_1 a point p where the curve is touched by the intersecting curve C_0; the locus of all points p is a common envelope Σ of the curves C_1 and C_0; in fact, when the curve C_1 by translation passes over into the next consecutive curve, the fixed point p moves along the curve C_0, that is along C_1 itself. The surface may therefore be conceived as generated by a translation of the curve C_1 in such a way that it always touches the envelop Σ, or by translation of C_0 in the same manner.* The curve Σ is an asymptotic line, since the tangents along it are conjugate to themselves.

Suppose, for example, that the translation-surface is represented by the equations

$$x = A(u) + A(v), \qquad y = B(u) + B(v), \qquad z = C(u) + C(v), \tag{2}$$

which may be obtained from (1) by letting A_1, B_1, C_1 be the same functions of (v) as A, B, C are of (u). In this case the above condition is satisfied, C_0 and C_1 being congruent. It is evident that the envelope Σ is obtained by putting $u = v$ in (2) so that the equation of this curve becomes

$$x = 2A(u), \qquad y = 2B(u), \qquad z = 2C(u),$$

which is an asymptotic line similar to C_0 and C_1 and drawn on twice the scale. *The surface* (2) *may therefore be considered as the locus of the middle points of all chords of* Σ; *the curves* C_0 *and* C_1 *form a single irreducible family of congruent and similarly placed curves.*

In 1872 Lie proposed and solved the problem to find all translation-surfaces

* For proof of this theorem see Lie-Scheffer, Berührungstransformationen, p. 362.

which may be generated by translation in an infinite number of ways.* These surfaces belong to a more general class that are generated by translation in four different ways. In 1882 Lie completely determined all such surfaces.† Later he noticed that this problem had some connection with Abel's theorem applied to curves of the fourth order, but it was not till 1892 that, after repeated efforts, he was able to state and prove the following important theorem:

If on a translation-surface that can be generated in more than two ways we draw tangents at any point along the four generating curves, the intersection of these tangents with the plane at infinity is a curve of the fourth order.

Conversely, if we suppose given in the plane at infinity a curve of the fourth order, there exist always infinitely many (∞^4) *surfaces generated in four ways, whose tangents along the generating curves intersect the plane at infinity in points situated on the given curve.*

The co-ordinates of these surfaces are expressible as a sum of any two Abelian Integrals with respect to the four points of intersection of any variable straight line with the quartic.

Every direction in space is determined by a point in the plane at infinity. Let x, y, z be a point in space and $x + dx, y + dy, z + dz$ the consecutive point; the direction of a line joining these two points is completely determined whenever the ratios $\frac{dx}{dz}, \frac{dy}{dz}$ are given; we may therefore with Lie consider these ratios as coordinates ξ, η in the plane at infinity.§

Suppose now given in this plane a quartic curve $F(\xi, \eta) = 0$; in order to determine the translation-surface, according to Lie's theorem, we form the Abelian integrals of the first kind

$$\Phi = \int \frac{\xi \, d\xi}{F'_{(\eta)}}, \qquad \Psi = \int \frac{\eta \, d\eta}{F'_{(\eta)}}, \qquad \mathrm{X} = \int \frac{d\xi}{F'_{(\eta)}}.$$

The limits of these integrals we fix in the following manner: We suppose the quartic cut by a fixed and a variable straight line; denoting the abscissas of the

* Kurzes Resumé mehrerer neuen Theorien, Ges. d. w. zu Kristiania, May 3d, 1872. Archiv for Math. og Naturvidenskab, B. 4, 1879.

† Archiv for Math. og Nat. B. 7, Kristiania 1882.

‡ Compte Rendue, B. 114 (1893, p. 334-337).

§ Lie-Scheffer, Berührtr. p. 357.

points of intersection by ξ_1^0, ξ_2^0, ξ_3^0, ξ_4^0, and ξ_1, ξ_2, ξ_3, ξ_4 respectively, we choose the former as the lower and the latter as the upper limits so that we have,

$$\Phi_i = \int_{\xi_i^0}^{\xi_i} \frac{\xi_i \, d\xi_i}{F'_{(\eta_i)}}, \quad \Psi_i = \int_{\xi_i^0}^{\xi_i} \frac{\eta_i \, d\eta_i}{F'_{(\eta_i)}}, \quad \mathrm{X}_i = \int_{\xi_i^0}^{\xi_i} \frac{d\xi_i}{F'_{(\eta^i)}}. \qquad (i = 1, 2, 3, 4.)$$

Now by Abel's theorem we have

$$\Phi_1 + \Phi_2 + \Phi_3 + \Phi_4 = 0,$$
$$\Psi_1 + \Psi_2 + \Psi_3 + \Psi_4 = 0,$$
$$\mathrm{X}_1 + \mathrm{X}_2 + \mathrm{X}_3 + \mathrm{X}_4 = 0,$$

from which it follows that

$$\Phi_1 + \Phi_2 = -\Phi_3 - \Phi_4,$$
$$\Psi_1 + \Psi_2 = -\Psi_3 - \Psi_4,$$
$$\mathrm{X}_1 + \mathrm{X}_2 = -\mathrm{X}_3 - \mathrm{X}_4,$$

so that the equations

$$x = \Phi_1 + \Phi_2, \qquad y = \Psi_1 + \Psi_2, \qquad z = \mathrm{X}_1 + \mathrm{X}_2 \tag{3}$$

represent the same surface as

$$x = -\Phi_3 - \Phi_4, \qquad y = -\Psi_3 - \Psi_4, \qquad z = -X_3 - X_4, \tag{4}$$

a translation-surface generated in four ways, as is seen from the double mode of representation.

If the quartic is irreducible, the integrals Φ_1, Φ_2, Φ_3, and Φ_4 have the same form, and likewise the Ψ's and X's; the surface (3), or (4), has therefore the same property as the surface (2), *i. e.* $\xi_1 = \xi_2$ is an envelope of the generating curves $\xi_1 =$ const., $\xi_2 =$ const., and $\xi_3 = \xi_4$ an envelope of the curves $\xi_3 =$ const., $\xi_4 =$ const. Moreover the surface is symmetrical with respect to the origin as is evident from equations (3) and (4); *it has therefore a centre.*

A linear projective transformation leaves the plane at infinity at rest; any curve in this plane will therefore be transformed projectively into a curve in the same plane, while the corresponding translation-surface will be transformed into a translation-surface. A given linear projective transformation of the surface determines a projective transformation of the curve $F(\xi, \eta) = 0$. If, on the other hand, a projective transformation of $F(\xi, \eta) = 0$ is given, the linear transformation of the surface is not uniquely determined; for, let the transformation be,

$$\xi_1 = \frac{a_1 \xi + a_2 \eta + a_3}{c_1 \xi + c_2 \eta + c_3}, \qquad \eta_1 = \frac{b_1 \xi + b_2 \eta + b_3}{c_1 \xi + c_2 \eta + c_3},$$

where

$$\xi_1 = \frac{dx_1}{dz_1}, \qquad \eta = \frac{dy_1}{dz_1}, \qquad \xi = \frac{dx}{dz}, \qquad \eta = \frac{dy}{dz},$$

and hence,

$$dx_1 = \rho(a_1\, dx + a_2\, dy + a_3\, dz),$$
$$dy_1 = \rho(b_1\, dx + b_2\, dy + b_3\, dz),$$
$$dz_1 = \rho(c_1\, dx + c_2\, dy + c_3\, dz),$$

from which we obtain the transformation

$$x_1 = \rho(a_1 x + a_2 y + a_3 z) + a_4,$$
$$y_1 = \rho(b_1 x + b_2 y + b_3 z) + b_4,$$
$$z_1 = \rho(c_1 x + c_2 y + c_3 z) + c_4,$$

where ρ_1, a_4, b_4, c_4 are entirely arbitrary, q. e. d. Thus, suppose given the identical transformation $\xi_1 = \xi$, $\eta_1 = \eta$, which leaves the quartic unaltered; we have,

$$\begin{array}{lll} a_1 = 1, & a_2 = 0, & a_3 = 0, \\ b_1 = 0, & b_2 = 1, & b_3 = 0, \\ c_1 = 0, & c_2 = 0, & c_3 = 1, \end{array}$$

and the corresponding transformation in space becomes

$$x_1 = \rho x + a_4, \qquad y_1 = \rho y + b_4, \qquad z_1 = \rho z + c_4;$$

hence, *to the same quartic curve in the plane at infinity belong* ∞^4 *translation-surfaces.*

All the translation-surfaces that are projectively equivalent with respect to a linear projective transformation are said to belong to the same *type*. If therefore we wish to obtain a certain type we consider all the projectively equivalent quartics in the plane at infinity and form the Abelian integrals with respect to all such curves.

The quartic curve in the plane at infinity may degenerate as follows:

(1) In a cubic and a straight line,

(2) In two conics,

(3) In a conic and two straight lines,

(4) In four straight lines.

The three last cases have been studied by R. Kummer in a thesis published in 1894.*

When the quartic degenerates into two conics two essentially different cases may present themselves.† Let the conics be C_1 and C_2 and let the variable line

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* Die Flächen mit unendlichvielen Erzeugenden durch Translation von Curven, Inaugural-Dissertation von Richard Kummer; Leipzig, 1894.

† See Lie-Scheffer, Berührtr. B. 1, p. 409.

cut them in four points P_1, P_2, P_3, P_4. If now p be a point on the surface, two things may happen:

1°. The point P_1 and P_2, viz., the points of intersection of the conjugate tangents $p\,P_1$, $p\,P_2$ with the plane at infinity, lie on the same conic, or,

2°. They lie on different conics.

In the first case it may be proved by means of Desargue's theorem that the corresponding surfaces contain ∞^1 congruent translation-curves corresponding to the family of ∞^1 conics determined by the two given conics;* *the corresponding surface is therefore generated in an infinite number of ways.*

In the second case we obtain surfaces generated in four ways only.† The general equation of these surfaces is

$$Ae^{z+y} + Be^{z+x} + Ce^{x+y} + Le^x + Me^y + Ne^z = 0. \tag{5}$$

To this class belong also all the transforms of (5) by linear transformations and their degenerates obtained by special choice of the conics C_1 and C_2.

In the first and more important case we obtain the surfaces

$$Ae^x + Be^y + Ce^z + D = 0 \tag{6}$$

and their transforms by linear projective transformations.‡

If one of the conics is the imaginary circle $\xi^2 + \eta^2 + 1 = 0$, Scherk's minimal surface

$$e^z = \frac{\sin x}{\sin y}$$

and its transforms are obtained; this corresponds to the case where the family of conics is determined by four imaginary points. If two of these points are real we obtain the transcendental surface

$$e^z = \frac{e^{2x} + 1}{-\,2e^x \sin y}.$$

If the four points are the I and J points counted twice, that is to say, the family of conics are concentric circles, we get the helicoid

$$z = \arctan \frac{y}{x}.$$

* For proof see Lie-Scheffer, Berührtr., p. 406.

† p. 364 ibid.

‡ By linear projective transformations we mean here as elsewhere the following:

$$x' = a_1x + b_1y + c_1z + d_1, \qquad y' = a_2x + b_2y + c_2z + d_2, \qquad z' = a_3z + b_3y + c_3z + d_3.$$

Finally, if all the four points coincide, forming a point of contact of the third order, Cayley's ruled surface is obtained.*

The case where the quartic degenerates into a cubic and a straight line has been treated in a thesis by Georg Wiegner (1893).† He arrives at the important result that *when the straight line is an inflexional tangent to the cubic the corresponding translation-surface has a family of parabolae as generators.* He tried to prove the converse of this theorem, but failed for the very good reason that it is not true. In fact, we shall prove that *in the case of a cuspidal cubic and its cuspidal tangent the same thing happens: A surface belongs to it which has a set of parabolae as generators and which is algebraic and of the fourth degree, while the only algebraic surface that Wiegner obtained is of the third degree.*‡

By a projective transformation a cuspidal cubic may be brought into the form

$$\eta^2 - \xi^3 = 0,$$

whose cuspidal tangent is $\eta = 0$. Putting now

$$F(\xi, \eta) = (\eta^2 - \xi^3)\,\eta = 0$$

and forming the Abelian integrals of the first kind with respect to this quartic, we have, since $F'_{\eta_1} = 2\xi_1^3$ and $F'_{\eta_2} = -2\xi_2^3$,

$$X = \tfrac{1}{2}\int \frac{\xi_1\, d\xi_1}{\xi_1^3} - \int \frac{\xi_2\, d\xi_2}{\xi_2^3} = -\tfrac{1}{2}\,\xi_1^{-1} + \xi_2^{-1} + \text{const.}$$

$$Y = \tfrac{1}{2}\int \frac{\xi_1^{3/2}\, d\xi_1}{\xi_1^3} - \int \frac{\eta_2\, d\xi_2}{\xi_2^3} = -\xi_1^{-\frac{1}{2}} + \text{const.},$$

$$Z = \tfrac{1}{2}\int \frac{d\xi_1}{\xi_1^3} - \int \frac{d\xi_2}{\xi_2^3} = -\tfrac{1}{4}\,\xi_1^{-2} + \tfrac{1}{2}\,\xi_2^{-2} + \text{const.}$$

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*Such is the case with all the conics of the family $\xi^2 + k\eta^2 + 3\eta = 0$, which touch at (0, 0) and have the same curvature at this point.

†Über eine besondere Klasse von Translationsflächen, Inaugural-Dissertation von Georg Wiegner, Separatausdruck aus dem Archiv for Math. og Naturv., B. 16, 1893.

‡This surface is $zx^2 - 2yz = zx$. Georg Scheffer in his admirable paper "Das Abel'sche Theorem und das Lie'sche Theorem über Translationsflächen" (Acta Math, Vol. 28) has repeated Wiegner's mistake when he says: "Und nur in diesem Fall (viz., a cubic and its inflexional tangent) treten Parabeln als erzeugende Curven auf." See p. 90, ibid.

Putting const. $=0$ and transforming, these equations may be written

$$X=\frac{1}{\xi_1}-\frac{2}{\xi_2},$$
$$Y=\frac{1}{\sqrt{\xi_1}},$$
$$Z=\frac{1}{\xi_1^2}-\frac{2}{\xi_2^2},$$

from which we obtain the quartic surface

$$2Z=Y^4+2Y^2X-X^2, \tag{7}$$

the generating parabolæ of which are all parallel to the parabola $X^2+2Z=0$. The surface may therefore be generated by letting the vertex of this parabola be translated along the quartic space curve

$$X=\frac{1}{\xi_1}, \qquad Y=\frac{1}{\sqrt{\xi_1}}, \qquad Z=\frac{1}{\xi_1^2},$$

the plane of the parabolæ being always kept parallel to the YZ-plane. The intersection of the surface with any plane $Z=c$ is a quartic curve which, when $Z=0$, degenerates into two parabolæ, $Y^4+2Y^2X-X^2=0$, tangents to each other at the origin, which thus becomes a saddle-point. (See plate.) The second pair of generating curves may now be obtained. We put

$$X=-\frac{1}{\xi_3}-\Phi_1,$$
$$Y=-\frac{1}{\sqrt{\xi_3}}-\Phi_2,$$
$$Z=-\frac{1}{\xi_3^2}-\Phi_3,$$

and substitute in the equation (7), which is satisfied if $\Phi_1=\Phi_2^2$ and $\Phi_3=\Phi_2^4$; hence we have as second mode of representation

$$X=-\frac{1}{\xi_3}-\frac{1}{\xi_4},$$
$$Y=-\frac{1}{\sqrt{\xi_3}}-\frac{1}{\sqrt{\xi_4}},$$
$$Z=-\frac{1}{\xi_3^2}-\frac{1}{\xi_4^2}.$$

If we put $\xi_3 = \xi_4$ we obtain the envelope $X^2 + 2Z = 0$ which, as was explained above, is an asymptotic curve on the surface.* It should also be noticed that the family of asymptotic curves on (7) may be obtained by quadrature according to Lie, since it admits of the following projective transformation into itself

$$\bar{X} = \lambda^2 X, \qquad \bar{Y} = \lambda Y, \qquad \bar{Z} = \lambda^4 Z.$$

We shall now restate Wiegner's theorem as follows:

When a quartic breaks down into either a cubic and its inflexional tangent, or a cubic and its cuspidal tangent, the corresponding translation-surfaces have a family of parabolae as generating curves.†

To this may also be added the following theorem, which from our standpoint is important:

There exist two types of algebraic surfaces corsesponding to

1°. *A cuspidal cubic and its cuspidal tangent.*

2°. *A cuspidal cubic and its inflexional tangent.*

When the quartic breaks down into two conics C_1 and C_2, and if we choose the points P_1 and P_2 on C_1 and C_2 respectively, we obtain, as was stated above, surfaces of the general form

$$Ae^{y+z} + Be^{z+x} + Ce^{x+y} + Le^x + Me^y + Ne^z = 0.$$

By special choice of the conics these surfaces may degenerate *and even become algebraic.* Two real conics being given, the following six cases may occur:

The two conics may intersect in

1°. Four real and distinct points.

2°. Two real and two imaginary points.

3°. Four imaginary points.

4°. Two pairs of consecutive points.

5°. Four real points of which three are consecutive.

6°. Four consecutive points.

In all cases except the last we obtain transcendental surfaces.

We shall not prove this proposition, as it involves rather tedious calculations, taking up one case after another and forming the corresponding Abelian integrals in each case. We shall therefore take up the last case and study the corresponding surfaces.

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* This curve is drawn on the model. See plate.

† The converse of this theorem has been omitted; it may easily be proved.

Let the conics C_1 and C_2 have four consecutive points in common. By a projective transformation we may always transform them in such a way that the point of contact coincides with the vertices of the conics. A family of such conics is represented by the equation

$$\eta^2 + k\xi^2 - \xi = 0.$$

Taking any two values of k, say k_1 and k_2, we write the quartic

$$F(\xi\eta) \doteq (\eta^2 + k_1\xi^2 - \xi)(\eta^2 + k_2\xi^2 - \xi) = 0,$$

or, homogeneously,

$$F(\xi, \eta, \zeta) = (\eta^2 + k_1\xi^2 - \xi\zeta)(\eta^2 + k_2\xi^2 - \xi\zeta) = 0,$$

which by means of the transformation

$$\eta = \eta', \qquad \xi = \xi', \qquad \zeta = \zeta' + \tfrac{1}{2}(k_1 + k_2)\xi'$$

may be brought into the form

$$[\eta^2 + \tfrac{1}{2}(k_1 - k_2)\xi^2 - \xi\zeta][\eta^2 - \tfrac{1}{2}(k_1 - k_2)\xi^2 - \xi\zeta] = 0.$$

If further we put $\sqrt{\tfrac{1}{2}(k_1 - k_2)}\,\xi = \xi'$, and $\zeta = \sqrt{\tfrac{1}{2}(k_1 - k_2)}\,\zeta'$, we obtain the following simple form for the two conics, putting $\zeta' = 1$:

$$F(\xi, \eta) = (\eta^2 + \xi^2 - \xi)(\eta^2 - \xi^2 - \xi) = 0.$$

We have now

$$F'_{\eta_1} = 4\xi_1^2\eta_1, \qquad F'_{\eta_2} = -4\xi_2^2\eta_2,$$

and forming the Abelian integrals we obtain the following parametric representation of the surface:

$$X = \tfrac{1}{4}\int\frac{d\xi_1}{\xi_1\sqrt{\xi_1^2 + \xi_1}} - \tfrac{1}{4}\int\frac{d\xi_2}{\xi_2\sqrt{\xi_2 - \xi_2^2}},$$

$$Y = \tfrac{1}{4}\int\frac{d\xi_1}{\xi_1^2} - \tfrac{1}{4}\int\frac{d\xi_2}{\xi_2^2},$$

$$Z = \tfrac{1}{4}\int\frac{d\xi_1}{\xi_1^2\sqrt{\xi_1^2 + \xi_1}} - \tfrac{1}{4}\int\frac{d\xi_2}{\xi_2^2\sqrt{\xi_2 - \xi_2^2}}.$$

Integrating and using the transformations

$$-4X = X', \qquad -4Y = Y', \qquad -4Z = Z', \qquad \frac{1}{\xi_1} = u^2 - 1, \qquad \frac{1}{\xi_2} = v^2 + 1,$$

we obtain

$$X = u - v, \qquad Y = u^2 - v^2 - 2, \qquad Z = u^3 - v^3 - 3u - 3v. \tag{8}$$

Eliminating u and v we obtain the surface

$$4ZX = X^4 + 3Y^2 - 12 = 0,$$

which may be easily thrown into the form

$$ZX = X^4 + Y^2 - 1, \tag{8'}$$

a quartic surface *which is symmetrical with respect to the origin.* The planes $X =$ const. are parabolic sections which for $X = 0$ degenerate into two straight lines $Y = \pm 1$. The locus of the vertices of these parabolæ is a plane quartic obtained by putting $Y = 0$ in (8′). *The generating curves are twisted cubics,* as is easily seen for equations (8).

To the same reducible quartics belongs also the surface obtained from (8′) by means of the imaginary transformations

$$X = iX', \qquad Y = iY', \qquad Z = iZ,$$

that is, the surface

$$ZX = -X^4 - Y^2 - 1,$$

which differs from (8′) by having two separate sheats, the plane $X = 0$ being asymptotic to the surface. We shall state the above result thus: *To a reducible quartic consisting of two conics having four consecutive points in common there corresponds a type of algebraic translation-surfaces, generated in four distinct ways, which are reducible to the form*

$$ZX = X^4 + Y^2 - 1.$$

II.

We shall now take up the study of algebraic surface connected with an irreducible quartic, and we propose to solve the problem of finding all the types of algebraic surfaces corresponding to such a curve. The results obtained will be interesting not only from the standpoint of surface-theory but also from that of theory of functions. The Abelian integral connected with a unicursal curve has naturally no proper period; it becomes what Poincaré calls a degenerate Abelian integral possessing poles and logarithmic singularities.* The following investigation may therefore be considered as a study of such integrals.

If the quartic is of genus 1, its co-ordinates are expressible as functions of a parameter θ and $\sqrt{\Theta}$ where Θ is of the fourth degree in θ. The co-ordinates

*Sur les surface de translation et les fonctions Abeliennes, Bulletin de la Societé Math., v. 29 (1901), pp. 61-86.

X, Y, Z of the corresponding surface are therefore elliptic integrals and the surface is transcendental. It is evident that quartics of genus 2 and 3 also give rise to the same kind of surfaces. We may therefore limit ourselves to unicursal curves, in which case the co-ordinates are expressible as rational functions of a parameter. The co-ordinates of the surface thus become integrals of rational functions. The problem is now to determine the cases in which these integrals themselves become algebraic functions. We shall prove the following important

Theorem. Whenever the unicursal quartic in the plane at infinity has at least one double point with distinct tangents the corresponding surface is transcendental.

Let the quartic have one double point and two cusps. By means of a proper projective transformation it may be thrown into the form

$$y^2 + x^2 - 2xy + x^2 y^2 - 2x^2 y + 2\,kxy^2 = 0, \tag{9}$$

having a cusp at the origin and at the end of the x-axis, while the double point is at the end of the y-axis. A parametric representation of this curve is obtained by determining each point on it as the intersection of the curve with a variable conic passing through the three singular points and another fixed point on the curve.* We shall choose as our conic the hyperbola

$$xy + \rho x + \sigma y = 0,$$

and as the fixed point the intersection of the cuspidal tangent at the origin with the curve. It is evident that the following relation between ρ and σ must exist:

$$\rho + \sigma = 2\,(k - 1).$$

Substituting now $x = \dfrac{-\sigma y}{\rho + y}$ in (9) and reducing, we easily find the co-ordinates of the variable point, viz:

$$x = \frac{2\,(1-k)}{(\rho+1)^2},$$

$$y = \frac{2\,(1-k)}{\rho^2 + 2\,(2-k)\,\rho + 5 - 4k},$$

which will serve as a parametric representation of the quartic. Substituting these values in

$$F'_y = 2\,(y - x + x^2 y - x^2 + 2kxy)$$

we obtain, after some reduction, the following simple expression:

$$F'_y = \frac{-8\,(1-k)}{(\rho+1)^3}.$$

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* See Salmon, Higher Plane Curves, p. 250.

We now form the Abelian integrals and obtain the surface

$$X=\int\frac{d\rho_1}{(\rho_1+1)^2}+\int\frac{d\rho_2}{(\rho_2+1)^2},$$
$$Y=\int\frac{d\rho_1}{\rho_1^2+2\,(2-k)\,\rho_1+5-4k}+\int\frac{d\rho_2}{\rho_2^2+2\,(2-k)\,\rho_2+5-4k},$$
$$Z=\frac{1}{2\,(1-k)}\int d\rho_1+\frac{1}{2\,(1-k)}\int d\rho_2,$$

or, performing the integration,

$$X=-\frac{1}{\rho_1+1}-\frac{1}{\rho_2+1},$$
$$Y=\frac{1}{\sqrt{1-k^2}}\tan^{-1}\frac{\rho_1+2-k}{\sqrt{1-k^2}}+\frac{1}{\sqrt{1-k^2}}\tan^{-1}\frac{\rho_2+2-k}{\sqrt{1-k^2}},$$
$$Z=\frac{1}{2\,(1-k)}\,\rho_1+\frac{1}{2\,(1-k)}\,\rho_2,$$

which is evidently a transcendental surface, provided k is an arbitrary constant differing from $+1$ or -1 Q. E. D.

When $k=1$ the quartic breaks down into two coincident hyperbolæ, hence this value of k must be excluded.

If $k=-1$ the curve becomes a tricuspidal quartic or *deltoid*;* we may therefore obtain the corresponding surface by putting $k=1$ in (9) and integrating. We thus obtain the algebraic surface

$$\begin{aligned} X&=-\frac{1}{\rho_1+1}-\frac{1}{\rho_2+1},\\ Y&=-\frac{1}{\rho_1+3}-\frac{1}{\rho_2+3},\\ Z&=\tfrac{1}{4}\rho_1+\tfrac{1}{4}\rho_2. \end{aligned}\tag{10}$$

Putting $X=-X'$, $Y=-Y'$, $4Z=Z'$ we obtain by eliminating ρ_1 and ρ_2

$$Z\,(2XY+Y-X)=6X-2Y-8XY;$$

and transforming this surface to the centre of symmetry which is $X=\frac{1}{2}$, $Y=-\frac{1}{2}$, $Z=4$, we reduce the equation to the simpler form

$$Z\,(2XY+\tfrac{1}{2})=2\,(X+Y)\tag{11}$$

which is a cubic surface *whose sections parallel to the three co-ordinate planes are*

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*I have called the corresponding surface *the deltoid surface.* I have made a model of this surface, which, together with others, are found in the collection of the Johns Hopkins University, Baltimore.

hyperbolae. In the finite portion of space the following right lines are seen to lie on the surface:

(1) $Z=0,\ X+Y=0$; (2) $Z=2,\ X=\frac{1}{2}$; (3) $Z=2,\ Y=\frac{1}{2}$;
(4) $X=0,\ Z=4Y$; (5) $Y=0,\ Z=4X$; (6) $X=\frac{1}{2},\ Y=-\frac{1}{2}$;
(7) $X=-\frac{1}{2},\ Y=\frac{1}{2}$, (8) $Z=-2,\ X=-\frac{1}{2}$; (9) $Z=-2,\ Y=-\frac{1}{2}$.

The locus of the vertices of the parabolae $Z=$ const. is a plane cubic. In order to find its equation it will be convenient to transform the X and Y-axis by turning them through an angle of 45°; we obtain then

$$Z(X^2-Y^2+\tfrac{1}{2})=\sqrt{2}X.$$

Putting $Y=0$, we have

$$Z(X^2+\tfrac{1}{2})=\sqrt{2}\,X,$$

a cubic curve having the X— axis as asymptote, and a maximum and minimum at $Z=\pm 1$, $X=\pm\frac{1}{2}\sqrt{2}$, corresponding to the two saddle-points on the surface.

The translation-curves are twisted cubics, as is easily seen from the parametric representation of the surface (11) which is

$$X=\frac{1}{\rho_1+1}+\frac{1}{\rho_2+1}+\tfrac{1}{2},$$
$$Y=\frac{1}{\rho_1+3}+\frac{1}{\rho_2+3}-\tfrac{1}{2},$$
$$Z=\rho_1+\rho_2+4.$$

The envelope of these translation-curves is

$$X=\frac{2}{\rho_1+1}+\tfrac{1}{2},$$
$$Y=\frac{2}{\rho_1+3}-\tfrac{1}{2},$$
$$Z=2\rho_1+4.$$

We shall state the results of the above development as follows:

*The necessary condition that a translation-surface that can be generated in four ways is algebraic, is that the corresponding quartic in the plane at infinity is unicursal and possesses no double point with distinct tangents.**

Before we discuss the cases where the unicursal quartic has a tac-node, osc-node, or triple point, we shall study a rather interesting case of the surface (11) belonging to the same type.

* The following investigations will show that this is also the sufficient condition.

If we apply to the quartic (9) the linear imaginary transformation

$$x = x' + iy', \qquad y = x' - iy'$$

we obtain the cardioid

$$(x^2 + y^2 - 2x)^2 = 4\,(x^2 + y^2).$$

This transformation places the two cusps at infinity at the imaginary I and J points. The parametric representation of this curve is

$$x = \frac{4\,(1-\rho^2)}{(\rho^2+1)^2}, \quad y = \frac{8\rho}{(\rho^2+1)^2}.$$

We have now

$$dx = \frac{8\rho\,(\rho^2-3)}{(\rho^2+1)^3}, \quad F'_y = \frac{64\rho\,(3-\rho^2)}{(1+\rho^2)^3},$$

and the equation of the corresponding surface is

$$X = -\tfrac{1}{2}\int\frac{(1-\rho_1^2)}{(1+\rho_1^2)}\,d\rho_1 - \tfrac{1}{2}\int\frac{(1-\rho_2^2)}{(1+\rho_2^2)^2}\,d\rho_2 = -\tfrac{1}{2}\frac{\rho_1}{1+\rho_1^2} - \tfrac{1}{2}\frac{\rho_2}{1+\rho_2^2},$$

$$Y = -\int\frac{\rho_1\,d\rho_1}{(1+\rho_1^2)^2} - \int\frac{\rho_2\,d\rho_2}{(1+\rho_2^2)^2} = -\frac{1}{1+\rho_1^2} - \frac{1}{1+\rho_2^2},$$

$$Z = -\tfrac{1}{8}\int d\rho_1 - \tfrac{1}{8}\int d\rho_2 = -\tfrac{1}{8}\rho_1 - \tfrac{1}{8}\rho_1.$$

Eliminating ρ_1 and ρ_2 we obtain the equation

$$4Z\,(4X^2 + Y^2 - Y) = X,$$

which by a suitable real and linear transformation may be thrown into the form

$$Z\,(X^2 + Y^2 - 1) = 2X.^* \tag{12}$$

A plane $Z =$ const. cuts out a circle whose equation is

$$\left(X - \frac{1}{c}\right)^2 + Y^2 = 1 + \frac{1}{c^2}.$$

For $Z = \infty$ we obtain the unit circle and for $Z = 0$ a straight line $X = 0$. The locus of centres of these circles is the hyperbolae $XZ = 1$; the plane $Y = 0$ intersects the surface in the cubic curve $Z\,(X^2 - 1) = 2X$ which has a point of inflexion at the origin and the Z-axis as asymptote. Any plane $Z = mX$ cuts the surface in a family of ∞^1 circles whose planes all pass through the Y-axis.

To the same cardioid also belongs the surface obtained from (12) by using the transformation

$$X = iX', \qquad Y = iY', \qquad Z = iZ';$$

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* This surface I have called *The Cardioid-Surface.* See Plate.

the resulting cubic surface

$$Z(X^2 + Y^2 + 1) + 2X = 0 \qquad (12')$$

differs from (12) in several respects, although it belongs to the same type. The plane $Z = \infty$ intersects it in the imaginary circle $X^2 + Y^2 + 1 = 0$, and the planes $Z = \pm 1$ is tangent to the surface at the two umbilical points $(-1, 0, 1)$, $(1, 0, -1)$.

In conclusion it should be noticed that all the surfaces belonging to this type have a real double point at infinity.

III.

Quartics with Node-Cusp and Ac-Node Cusp.

We shall now consider the case where the quartic has an ordinary cusp and a tac-node. We shall prove the following theorem:

In order that the surface shall be algebraic, the tac-node must degenerate into a node-cusp.

To a quartic with a node-cusp and an ordinary cusp corresponds an algebraic translation-surface of the fourth degree.

A quartic with a tac-node and a cusp may by a projective transformation be thrown into the form

$$x^3 y + ax^4 + (1 - a) yx^2 - y^2 = 0,$$

in which the tac-node is placed at the origin and the cusp at the end of the y-axis. For the sake of convenience we shall put $a = b - 1$, so that the parametric representation may be written

$$x = \frac{\rho^2 + b\rho}{\rho + 1}, \quad y = \frac{(\rho^2 + b\rho)^2}{\rho + 1}.$$

We have also,

$$F'_y = x^3 + (2 - b)x^2 - 2y = -\frac{(\rho^2 + b\rho)^2}{(\rho + 1)^3}[\rho^2 + (2 - b)\rho + b]$$

and

$$dx = \frac{\rho^2 + (2 - b)\rho + b}{(\rho + 1)^2}.$$

Forming now the Abelian integrals we have

$$X = -\int \frac{d\rho_1}{\rho_1^2 + b\rho_1} - \int \frac{d\rho_2}{\rho_2^2 - b\rho_2},$$

$$Y = -\int d\rho_1 - \int d\rho_2,$$

$$Z = -\int \frac{(\rho_1 + b)\, d\rho_1}{(\rho_1^2 + b\rho_1)^2} - \int \frac{(\rho_2 + b)\, d\rho_2}{(\rho_2^2 + b\rho_2)^2},$$

or, integrating,

$$X = -\frac{1}{b}\log\frac{\rho_1}{\rho_1+1} - \frac{1}{b}\log\frac{\rho_2}{\rho_2+1},$$
$$Y = -\rho_1 - \rho_2,$$
$$Z = -\frac{1}{b^2}\left(\frac{1}{\rho_1}+\frac{1}{\rho_2}\right) + \frac{b-2}{b^3}\log\frac{\rho_1\rho_2}{(\rho_1+1)(\rho_2+1)},$$

which is a transcendental surface for $b =$ any arbitrary constant differing from 0. Q. E. D.

If $b = 0$ the quartic reduces to the form

$$x^3y - x^4 + 2yx^2 - y^2 = 0,$$

which has a node-cusp at the origin. The corresponding surface is

$$\begin{aligned} X &= -\frac{1}{\rho_1} - \frac{1}{\rho_2}, \\ Y &= -\rho_1 - \rho_2, \\ Z &= -\tfrac{1}{2}\left(\frac{1}{\rho_1^2}+\frac{1}{\rho_2^2}\right) - \tfrac{1}{3}\left(\frac{1}{\rho_1^3}+\frac{1}{\rho_2^3}\right), \end{aligned} \tag{13}$$

or, eliminating,

$$Y(3X^2 + 2X^3 - 6Z) + 6X^2 + 6X = 0.$$

Transforming this surface to its centre of symmetry $(-\tfrac{1}{2}, 0, \tfrac{1}{12})$, we have

$$Y(2X^3 - \tfrac{3}{2}X - 6Z) + 6(X^2 - \tfrac{1}{4}) = 0,$$

which may be still further simplified by putting $\tfrac{2}{3}Y = Y'$, $X = X'$, $3X + 12Z = 4Z'$, so that we finally obtain

$$Y(4X^3 - 4Z) + 4X^2 - 1 = 0, \tag{13'}$$

which is a quartic surface whose plane sections parallel to the YZ-plane are equilateral hyperbolae. The translation-curves are twisted quartics as are also their two envelopes $\rho_1 = \rho_2$, $\rho_3 = \rho_4$. For $X = \pm\tfrac{1}{2}$ the sections become two pairs of intersecting lines. We shall not enter into a detailed study of this surface, but proceed to the next case.

When three consecutive nodes coincide, we have what is called an osc-node. If we consider such a node as formed by the coincidence of a tac-node with a double point, it will seem evident from the principle of continuity and from the preceding development that also in this case will the surface be transcendental. We shall, however, prove this more rigorously thus:

A quartic with an osc-node at the origin may be written (See Salmon, Higher Plane Curves, p. 249, sec. ed., § 290)

$$(y - mx^2)^2 + cxy(y - x^2) + dy^3 + gx^2y^2 + hxy^3 + iy^4 = 0,$$

or, introducing homogeneous co-ordinates,

$$(yz - mx^2)^2 + cxy(yz - x^2) + dy^3z + gx^2y^2 + hxy^3 + iy^4 = 0.$$

We now choose the vertices of the triangle of reference as follows: Let $x = 0$, $y = 0$ be the osc-node, $y = 0$ being the tangent at this point. As second vertex, $y = 0$, $z = 0$, we take a point of inflexion on the curve, making $z = 0$ the inflexional tangent. The third vertex will then be uniquely determined at some point which in general does not lie on the curve. The equation will now have the form

$$(yz - mx^2)^2 + cxy(yz - mx^2) + dy^3z = 0,$$

in which the coefficients d and m may be reduced to unity by putting $x = \frac{x'}{\sqrt{dm}}$, $y = \frac{y'}{d}$, $z = z'$, so that we finally obtain the simple form

$$(y - x^2)^2 + cxy(y - x^2) + y^3 = 0.* \tag{14}$$

A parametric representation of this curve may now be obtained as follows:

Writing (14) in the form

$$(y - x^2)^2 + cxy(y - x^2) + y^2(y - x^2) + x^2y^2 = 0,$$

we see that the curve is closely connected with the conic

$$z'^2 + cx'z' + y'z' + x'^2 = 0,†$$

which may be represented parametrically by putting

$$x' = -\sigma\left(t + \frac{c}{2}\right), \quad y' = \sigma\left(\frac{c^2}{4} - 1 - t^2\right), \quad z' = \sigma;$$

but since $x' : y' : z' = xy : y^2 : y - x^2$, we have

$$xy = -\sigma\left(t + \frac{c}{2}\right), \quad y^2 = \sigma\left(\frac{c^2}{4} - 1 - t^2\right), \quad y - x^2 = \sigma.$$

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*If dxm have opposite signs we put $x = \frac{x'}{\sqrt{-gm}}$, $y = \frac{y}{d}$, $z = z'$ which reduces the quartic to the form

$$(y + x^2)^2 + cxy(y + x^2) + y^3 + 0.$$

†See Salmon, Higher Plane Curves, p. 249 sec. ed.

Solving these equations for x and y we find

$$x = \frac{\left(t + \frac{c}{2}\right)\left(t^2 + \frac{c^2}{4} - 1\right)}{ct + \frac{c^2}{2} - 1}, \qquad y = \frac{\left(t^2 - 1 + \frac{c^2}{4}\right)^2}{ct + \frac{c^2}{2} - 1}.$$

Calculating now F'_y in terms of t and forming the Abelian integrals we have the surface

$$\begin{aligned}
X &= -\int \frac{\left(t_1 + \frac{c}{2}\right) dt_1}{\left(\frac{c^2}{4} - 1\right)^2 - t_1^4} - \int \frac{\left(t_2 + \frac{c}{2}\right) dt_2}{\left(\frac{c^2}{4} - 1\right)^2 - t_2^4}, \\
Y &= -\int \frac{dt_1}{\frac{c^2}{4} - 1 - t_1^2} - \int \frac{dt_2}{\frac{c^2}{4} - 1 - t_2^2}, \\
Z &= \int \frac{\left(ct_1 + \frac{c^2}{2} - 1\right) dt_1}{\left(\frac{c^2}{4} - 1 + t_1^2\right)^2 \left(\frac{c^2}{4} - 1 - t_1^2\right)} - \int \frac{\left(ct_2 + \frac{c^2}{2} - 1\right) dt_2}{\left(\frac{c^2}{4} - 1 + t_2^2\right)^2 \left(\frac{c^2}{4} - 1 - t_2^2\right)},
\end{aligned} \tag{15}$$

which is transcendental provided c differs from 2 or -2. Q. E. D.

If $c = \pm 2$ the surface (15) is algebraic and only then. In these cases the quartic has a tac-node-cusp at the origin. Now the case $c = -2$ may be reduced to that of $c = +2$ by a projective transformation. We have therefore the theorem:

To a quartic with a tac-node-cusp corresponds the algebraic surface

$$\begin{aligned}
X &= -\int \frac{t_1 + 1}{t_1^3} dt_1 - \int \frac{t_2 + 1}{t_2^4} dt_2 = \tfrac{1}{2}\left(\frac{1}{t_1^2} + \frac{1}{t_2^2}\right) + \tfrac{1}{3}\left(\frac{1}{t_1^3} + \frac{1}{t_2^3}\right), \\
Y &= -\int \frac{dt_1}{t_1^2} - \int \frac{dt_2}{t_2^2} = \frac{1}{t_1} + \frac{1}{t_2}, \\
Z &= -\int \frac{2t_1 + 1}{t_1^6} dt_1 - \int \frac{2t_2 + 1}{t_2^6} dt_2 = \tfrac{1}{2}\left(\frac{1}{t_1^4} + \frac{1}{t_2^4}\right) + \tfrac{1}{5}\left(\frac{1}{t_1^5} + \frac{1}{t_2^5}\right),
\end{aligned}$$

and its transforms by linear projective transformations. If now we put $\frac{1}{t_1} = u$, $\frac{1}{t_2} = v$ and interchange Y and Z we have

$$\begin{aligned}
X &= \tfrac{1}{2}(u^2 + v^2) + \tfrac{1}{3}(u^3 + v^3), \\
Y &= \tfrac{1}{2}(u^4 + v^4) + \tfrac{1}{5}(u^5 + v^5), \\
Z &= u + v,
\end{aligned}$$

or, eliminating,

$$(1+Z)Y=(1+Z)\left(\frac{Z^5}{5}+\frac{Z^4}{2}\right)-(Z^3+2Z^2)\left(\frac{Z^2}{2}+\frac{Z^3}{3}-X\right)+\left(\frac{Z^2}{2}+\frac{Z^3}{3}-X\right)^2.$$

Transforming this surface to its centre of symmetry $(\frac{1}{3}, -\frac{1}{5}, 1)$ we have

$$ZY+\frac{1}{45}Z^6-\frac{1}{12}Z^4-\frac{1}{3}XZ^3-X^2+XZ+\frac{1}{2}Z^2+\frac{13}{36}=0,$$

which may be simplified by putting

$$\tfrac{1}{2}Z+Y+X=Y', \qquad X=X' \qquad Z=Z',$$

so that we have

$$ZY+\frac{1}{45}Z^6-\frac{1}{12}Z^4-\frac{1}{3}XZ^3-X^2+\frac{13}{36}=0, \tag{16}$$

a sextic translation-surface whose plane sections parallel to the XY-plane are parabolae, which for $Z=0$ become two parallel straight lines $X=\pm\sqrt{\frac{13}{36}}$; the surface resembles therefore (8′). The translation-curves are twisted quintics. The surface is unique in this respect, that its degree is the highest of any algebraic translation-surface of the kind we are considering.

IV.

Quartics with a Triple-Point.

A quartic with a triple-point may be written

$$z(ay^3+by^2x+cyx^2+dx^3)=kx^4+lx^3y+mx^2y^2+nxy^3+py^4.$$

We shall suppose that the tangents at the triple-point are all real and distinct. For the sake of convenience we shall place the triple-point at $z=0$, $x=0$ and then write the equation in non-homogeneous form

$$y(a+bx+cx^2+dx^3)=kx^4+lx^3+mx^2+nx+p.$$

Now, since $F'_y=a+bx+cx^2+dx^3$, we have

$$\begin{aligned}
X&=\int\frac{x_1\,dx_1}{a+bx_1+cx_1^2+dx_1^3}+\int\frac{x_2\,dx_2}{a+bx_2+cx_2^2+dx_2^3},\\
Y&=\int\frac{(kx_1^4+lx_1^3+mx_1^2+nx_1+p)\,dx_1}{(a+bx_1+cx_1^2+dx_1^3)^2}+\int\frac{(kx_2^4+lx_2^3+mx_2^2+nx_2+p)\,dx_2}{(a+bx_2+cx_2^2+dx_2^3)^2},\\
Z&=\int\frac{dx_1}{a+bx_1+cx_1^2+dx_1^3}+\int\frac{dx_2}{a+bx_2+cx_2^2+dx_2^3},
\end{aligned} \tag{17}$$

which is a transcendental surface. If, however, the tangents at the triple-point

coincide, this surface becomes algebraic and only in this case, as is easily seen, by putting $a + bx + cx^2 + dx^3 = d(x - \alpha)^3$ in (17). We may therefore say

To a quartic having a triple-point with coincident tangents correspond algebraic translation-surfaces.

We shall now consider these surfaces and find the simplest forms to which they can be reduced.

We write the quartic

$$yx^3 = kx^4 + lx^3 z + mx^2 z^2 + nxz^3 + pz^4;$$

interchanging the x and z-coordinates this may be wri ten non-homogeneously

$$y = k + lx + mx^2 + nx^3 + px^4.$$

By a projective transformation this equation may be reduced to one of the following two forms:

$$\begin{aligned} &(a) \quad y = x^2 + x^4, \\ &(b) \quad y = x^4. \end{aligned} \tag{18}$$

Since the form (a) is not projectively equivalent to (b), *there will be two types of surfaces,* the second type being connected with a quartic which in addition to the triple-point also has a point of undulation. We shall consider the case (a) first. The corresponding surface may be written

$$\begin{aligned} (2) \quad X &= \int x_1\,dx_1 + \int x_2\,dx_2 = x_1^2 + x_2^2, \\ Y &= \int (x_1^2 + x_1^4)\,dx_1 + \int (x_2^2 + x_2^4)\,dx_2 = \tfrac{1}{3}(x_1^3 + x_2^3) + \tfrac{1}{5}(x_1^5 + x_2^5), \\ Z &= \int dx_1 + \int dx_2 = x_1 + x_2. \end{aligned}$$

By elimination and transformation we get

$$Y = \frac{Z}{5}\left(\frac{5X^4 - Z^4}{4}\right) + \frac{Z}{6}(3X - Z^2),$$

whose centre of symmetry is easily found to be (1, 0, 0). Transforming to this point as origin we have,

$$Y = \frac{ZX^2}{4} - \frac{Z^3}{6} - \frac{Z^5}{20},$$

which by a homothetic transformation may be thrown into the form

$$Y = ZX^2 - Z^5 - Z^3, \tag{19}$$

a quintic surface resembling the surface (12) very closely, the difference being

that instead of having circular section parallel to the XY-plane it has parabolic sections. To the same curve belong also the real surfaces obtained from (19) by means of the imaginary transformation $X = iX'$, $Y = iY'$, $Z = iZ'$; we obtain the surface

$$Y = Z^3 - Z^5 - ZX^2. \tag{19'}$$

The intersection of the plane $Y = 0$ with the surface (19) is the line $Z = 0$ and the quartic $X^2 = Z^4 + Z^2$, which has a double point at the origin, the two branches going to infinity in all four quadrants. In (19′) the same plane cuts out $X^2 = Z^2 - Z^4$, which is a lemniscate. The curves cut out by the plane $X = 0$ are, in both cases quintic curves with a point of inflexion at the origin; but in the case of the surface (19) this quintic intersects the Z-axis in two real points outside of the origin and symmetrically situated with respect to it, while in the case of (19′) these two points are imaginary.

Case (b). The quartic $y = x^4$.
We have the surface

$$\begin{aligned}(2)\quad X &= \int x_1\,dx_1 + \int x_2\,dx_2 = x_1^2 + x_2^2,\\ Y &= \int x_1^4\,dx_1 + \int x_2^4\,dx_2 = \tfrac{1}{5}\,(x_1^5 + x_2^5)\,,\\ Z &= \int dx_1 + \int dx_2 = x_1 + x_2\,,\end{aligned}$$

or, transforming and eliminating,

$$Y = \frac{Z}{5}\,(5X^2 - Z^4), \tag{20}$$

which by a homothetic transformation may be reduced to the form

$$Y = ZX^2 - Z^5.$$

The locus of vertices of the parabolic sections is the curve $Y = -Z^5$, and the plane $Y = 0$ cuts out the straight line $Z = 0$ and two parabolae $X = \pm Z^2$. The surface resembles (19); its transform by the imaginary transformation used above intersects the plane $Y = 0$ in two imaginary parabolae. It should also be noticed that the asymptotic lines of (20) may be found by quadratures, according to a general theorem of Lie.

We shall now give a resumé of the preceding results as follows:

PLANE AT INFINITY, $(x\ y)$	SPACE OF THREE DIMENSIONS, (X, Y, Z)
Quartic Curve.	*Translation-surfaces that can be Generated in Four Distinct Ways.*
I. Two Conics: $(y^2+x^2-x)(y^2-x^2-x)=0$	I. $ZX = X^4 + Y^2 - 1$
II. A unicursal cubic and its inflexional tangent: $(y-x^3)\,y=0$	II. $ZX^2 - 2YZ = 2X$ (Wiegner's surface)
III. A unicursal cubic and its cuspidal tangent: $(y^2-x^3)\,y=0$	III. $2Z = Y^4 + 2Y^2X - X^2$
IV. Non-reducible unicursal quartics: (*a*) tricuspidal quartic, (*deltoid*) (*b*) *cardioid*	IV. *The deltoid surface*: (*a*) $Z(2YZ+\frac{1}{2}) = 2(X+Y)$ (*b*) *The cardioid surface*: $Z(X^2+Y^2-1)=2X$
V. Quartic with a node cusp and an ordinary cusp: $(x^3y - x^4 + 2yx^2 - y^2 = 0)$	V. $Y(4X^3 - 4Z) + 4X^2 - 1 = 0$
VI. Quartic with a tac-node cusp: $[(y-x^2)^2 + 2xy(y-x^2) + y^3 = 0]$	VI. $ZY + \frac{1}{45}Z^6 - \frac{1}{12}Z^4 - \frac{1}{3}XZ^3 - X^2 + \frac{13}{36} = 0$
VII. Quartic having a triple-point with coincident tangents: $(y = x^2 + x^4)$	VII. $Y = ZX^2 - Z^5 - Z^3$
VIII. Quartic having a triple-point with coincident tangents and a point of undulation: $(y = x^4)$	VIII. $Y = ZX^2 - Z^5$

These then are the eight different types of algebraic translation-surfaces that can be generated in four different ways. The preceding developments show that these are all the types of algebraic surfaces.* In a subsequent paper I intend to discuss the types of translation-surfaces other than algebraic connected with a unicursal quartic.

U. S. NAVAL ACADEMY, *May* 21, 1906.

* We have excluded here Cayly's ruled minimal surface mentioned above.